Upper limits of Sinai's walk in random scenery

by

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Summary. We consider Sinai's walk in i.i.d. random scenery and focus our attention on a conjecture of Révész concerning the upper limits of Sinai's walk in random scenery when the scenery is bounded from above. A close study of the competition between the concentration property for Sinai's walk and negative values for the scenery enables us to prove that the conjecture is true if the scenery has "thin" negative tails and is false otherwise.

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1 Introduction

1.1 Random walk in random environment

Problems involving random environments arise in different domains of physics and biology. Originally, one-dimensional random walk in random environment appeared as a simple model for DNA transcription. In the following, we consider the elementary model of one-dimensional Random Walk in Random Environment (RWRE), defined as follows. Let

$\omega := (\omega_i,\, i \in \mathbb{Z})$ be a family of independent and identically distributed (i.i.d.) random variables defined on Ω, which stands for the random environment. Denote by P the distribution of ω and by E the corresponding expectation.

Conditioning on ω (i.e., choosing an environment), we define the RWRE $(X_n,\, n \geq 0)$ as a nearest-neighbor random walk on $\mathbb{Z}$ with transition probabilities given by ω: $(X_n,\, n \geq 0)$ the Markov chain satisfying $X_0 = 0$ and for $n \geq 0$,

$$P_\omega\{X_{n+1} = x+1 \,|\, X_n = x\} = \omega_x = 1 - P_\omega\{X_{n+1} = x-1 \,|\, X_n = x\}.$$

We denote by P_ω the law of $(X_n,\, n \geq 0)$, by E_ω the corresponding expectation, and by $\mathbb{P}$ the joint law of $(\omega, (X_n)_{n\geq 0})$. We refer to Zeitouni [25] for an overview of random walks in random environment.

Throughout the paper, we make the following assumptions on ω:

$$\exists\, \delta \in (0, 1/2): \qquad P\{\delta \leq \omega_0 \leq 1-\delta\} = 1, \tag{1.1}$$

$$E\Big[\log\Big(\frac{1-\omega_0}{\omega_0}\Big)\Big] = 0, \tag{1.2}$$

$$\sigma^2 := \mathrm{Var}\Big[\log\Big(\frac{1-\omega_0}{\omega_0}\Big)\Big] > 0. \tag{1.3}$$

Assumption (1.1) implies that $|\log(\frac{1-\omega_0}{\omega_0})|$ is, P-a.s., bounded by the constant $L := \log(\frac{1-\delta}{\delta})$. It is a technical assumption, which can be replaced by an exponential moment of $\log(\frac{1-\omega_0}{\omega_0})$. According to a recurrence-transience result due to Solomon [24], assumption (1.2) ensures that $(X_n)_{n\geq 0}$ is $\mathbb{P}$-almost surely recurrent, i.e., the random walk hits any site infinitely often. Assumption (1.3) excludes the case of deterministic environment, which corresponds to the homogeneous symmetric random walk.

Under assumptions (1.1)–(1.3), the RWRE is referred to as Sinai's walk. Sinai [23] proves that $X_n/(\log n)^2$ converges in law, under $\mathbb{P}$, toward a non-degenerate random variable, whose distribution is explicitly computed by Kesten [17] and Golosov [10]. This result contrasts with the usual central limit theorem which gives the convergence in law of $X_n/\sqrt{n}$.

Let

$$\begin{aligned} L(n,x) &:= \#\{0 \leq i \leq n:\ X_i = x\}, \qquad n \geq 0,\ x \in \mathbb{Z}, \\ L(n,A) &:= \sum_{x\in A} L(n,x), \qquad n \geq 0,\ A \subset \mathbb{Z}. \end{aligned} \tag{1.4}$$

In words, the quantity $L(n,A)$ measures the number of visits to the set A by the walk in the first n steps.

The maximum of local time is studied by Révész ([20], p. 337) and Shi [21]: under assumptions (1.1)–(1.3), there exists $c_0 > 0$ such that

$$\limsup_{n\to\infty} \frac{\max_{x\in\mathbb{Z}} L(n,x)}{n} \geq c_0, \qquad \mathbb{P}\text{-a.s.}$$

It means that the walk spends, infinitely often, a positive part of its life on a single site. The liminf behavior is analyzed by Dembo, Gantert, Peres and Shi [6], who prove that

$$\liminf_{n\to\infty} \frac{\max_{x\in\mathbb{Z}} L(n,x)}{n/\log\log\log n} = c_0', \qquad \mathbb{P}\text{-a.s.},$$

for some $c_0' \in (0,\infty)$. A concentration property is obtained by Theorem 1.3 of Andreoletti [1], which says that, under assumptions (1.1)–(1.3) and for any $0 < \beta < 1$, there exists $\ell(\beta) > 0$ such that

$$\limsup_{n\to\infty} \frac{\sup_{x\in\mathbb{Z}} L\left(n, [x-\ell(\beta), x+\ell(\beta)]\right)}{n} \geq \beta, \qquad \mathbb{P}\text{-a.s.} \tag{1.5}$$

In words, for any β close to 1, it is possible to find a length $\ell(\beta)$ such that, $\mathbb{P}$-almost surely, the particle spends, infinitely often, more than a β-fraction of its life in an interval of length $2\ell(\beta)$.

1.2 Random walk in random scenery

Random Walk in Random Scenery (RWRS) is a simple model of diffusion in disordered media, with long-range correlations. It is a class of stationary random processes exhibiting rich behavior. It can be described as follows: given a Markov chain on a state space, there may be a random field indexed by the state space, called a random scenery. As the random walk moves on this state space, he observes the scenery at his location. For a survey of recent results about RWRS, we refer to den Hollander and Steif [14], and to Asselah and Castell [2] for large deviations results in dimension $d \geq 5$.

Let us now define the model of one-dimensional RWRS: consider $S = (S_n,\, n \geq 0)$ a random walk on $\mathbb{Z}$ and $\xi := (\xi(x),\, x \in \mathbb{Z}) = (\xi_x,\, x \in \mathbb{Z})$, a family of i.i.d. random variables defined on a probability space Ξ. We refer to ξ as the random scenery and denote by Q its law. Then, define the process $(Y_n,\, n \geq 0)$ by

$$Y_n := \sum_{i=0}^{n} \xi(S_i),$$

called RWRS or the Kesten-Spitzer Random Walk in Random Scenery. An interpretation is the following: if a random walker has to pay ξ_x each time he visits x, then Y_n stands for the total amount he has paid in the time interval $[0, n]$.

The model is introduced and studied by Kesten and Spitzer [18] in dimensions $d = 1$ and $d \geq 3$. They prove in dimension $d = 1$ that, when S and ξ belong to the domains of attraction of stable laws of indice α and β respectively, then there exists δ, depending on α and β, such that $n^{-\delta} Y_{\lfloor nt \rfloor}$ converges weakly. In the simple case where $\alpha = \beta = 2$, they show that

$$\left(n^{-3/4} Y_{\lfloor nt \rfloor};\, 0 \leq t \leq 1\right) \xrightarrow{\text{law}} (\Lambda(t);\, 0 \leq t \leq 1)\,,$$

where " $\xrightarrow{\text{law}}$ " stands for weak convergence in law (in some functional space; for example in the space of bounded functions on $[0, 1]$ endowed with the uniform topology). The process $(\Lambda(t),\, t \geq 0)$, called Brownian motion in Brownian scenery, is defined by $\Lambda(0) = 0$ and $\Lambda(t) := \int_{\mathbb{R}} \ell(t, x)\, \mathrm{d}W(x)$ for $t > 0$, where $(W(x);\, x \in \mathbb{R})$ denotes a two-sided Brownian motion and $(\ell(t, x),\, t \geq 0, x \in \mathbb{R})$ denotes the jointly continuous version of the local time process of a Brownian motion $(B(t),\, t \geq 0)$, independent of $(W(x);\, x \in \mathbb{R})$.

Independently, Borodin analyzes the case of one-dimensional nearest-neighbor random walk in random scenery, see [4] and [5]. Bolthausen [3] studies the case $d = 2$. He proves that, if S is a recurrent random walk and ξ_0 has zero expectation and finite variance, then $(n \log n)^{-1/2}\, Y_{\lfloor nt \rfloor}$ satisfies a functional central limit theorem.

1.3 Random environment and random scenery

In this paper, we consider Sinai's Walk in Random Scenery. Problems combining random environment and random scenery have been examined for more general models. Replacing $\mathbb{Z}$ by a more general countable state space, Lyons and Schramm [19] exhibit, under certain conditions, a stationary measure for Random Walks in a Random Environment with Random Scenery (RWRERS) from the viewpoint of the random walker. Häggström [11], Häggström and Peres [12] treat the case where the scenery arises from percolation on a graph. In this particular case, the scenery determines the random environment of the associated RWRE, which is used by the authors to obtain information about the scenery.

Let us first describe the model of Sinai's walk in random scenery. We consider Sinai's walk $(X_n,\, n \geq 0)$ under assumptions (1.1)–(1.3), and recall that the environment ω is defined on (Ω, P). For the scenery, we consider a family of i.i.d. random variables $\xi := (\xi(x),\, x \in \mathbb{Z}) = (\xi_x,\, x \in \mathbb{Z})$, defined on (Ξ, Q), independent of ω and $(X_n,\, n \geq 0)$. To translate independence

between ω and ξ, we consider the probability space $(\Omega\times\Xi, P\otimes Q)$, on which we define (ω,ξ). Moreover, we denote by $\mathbb{P}\otimes Q$ the law of $(\omega,(X_n)_{n\geq 0},\xi)$. Then we define as Sinai's walk in random scenery the process $(Z_n,\, n\geq 0)$:

$$Z_n := \sum_{i=0}^{n} \xi(X_i).$$

Observe that Z_n can be written using local time notation:

$$Z_n = \sum_{x\in\mathbb{Z}} \xi(x)\, L(n,x), \qquad n\geq 0, \tag{1.6}$$

where $L(n,x)$ stands for the local time of the random walk at site x until time n, see (1.4).

We are interested in the upper limit of Z_n in the case where $a := \operatorname{ess\,sup}\xi_0$ is finite. We consider the concentration property of order β for Sinai's walk with β close to 1 (see (1.5)), which enables us to formulate the conjecture of Révész ([20], p. 353): does the assumption that $a := \operatorname{ess\,sup}\xi_0$ is finite imply that, $\mathbb{P}\otimes Q$-almost surely,

$$\limsup_{n\to\infty}\frac{Z_n}{n} = a\,?$$

It turns out that the conjecture holds only under some additional assumptions on the distribution of the random scenery. It is interesting to note that this conjecture follows immediately from the result of Andreoletti [1] mentioned earlier if $\operatorname{ess\,inf}\xi_0$ is larger than $-\infty$. In the general case, a close study of the competition between the concentration property for Sinai's walk and negative values for the scenery enables us to obtain the following theorem, which gives a solution to this problem, depending on the tail decay of $\xi_0^- := \max\{-\xi_0, 0\}$.

Theorem 1.1 *Assume (1.1)–(1.3) and* $a := \operatorname{ess\,sup}\xi_0 < \infty$.

(i) If $Q\{\xi_0^- > \lambda\} \leq \frac{1}{(\log\lambda)^{2+\varepsilon}}$, *for some* $\varepsilon > 0$ *and all large* λ, *then*

$$\mathbb{P}\otimes Q\left\{\limsup_{n\to\infty}\frac{Z_n}{n} = a\right\} = 1.$$

(ii) If $Q\{\xi_0^- > \lambda\} \geq \frac{1}{(\log\lambda)^{2-\varepsilon}}$, *for some* $\varepsilon > 0$ *and all large* λ, *then*

$$\mathbb{P}\otimes Q\left\{\lim_{n\to\infty}\frac{Z_n}{n} = -\infty\right\} = 1.$$

It is possible to give more precision in the case (ii), see Remark 3.5. On the other hand, the case $\varepsilon = 0$ is still open.

We mention that, under (1.1)–(1.3) and $a := \operatorname{ess\,sup} \xi_0 < \infty$, it is possible to prove that $\mathbb{P} \otimes Q\{\limsup_{n\to\infty} \frac{Z_n}{n} = c\}$, satisfies a $0-1$ law, for any $c \in [-\infty, \infty]$. The proof follows the lines of [8], except that we need an additional argument saying that modifying a finite number of random variables in the scenery does not change the behavior of $\limsup_{n\to\infty} \frac{Z_n}{n}$. The latter can be done by means of Theorem 1 (see Example 1) in [16], which implies, for any $x \in \mathbb{Z}$, that $L(n,x) = o(n)$, $n \to \infty$, $\mathbb{P}$-almost surely.

In general, we do not know whether $\limsup_{n\to\infty} \frac{Z_n}{n} \in \{-\infty, a\}$, $\mathbb{P}$-almost surely.

The paper is organized as follows: in Section 2, we present some key results for the environment and for Sinai's walk when the environment is fixed (i.e., quenched results). In Section 3, we define precisely the notion of "good" environment-scenery and prove Theorem 1.1 by accepting two intermediate propositions. The first one, proved in Section 4, is devoted to the study of the RWRE within the "good" environment-scenery. The second one, proved in Section 5, does not concern the RWRE, but only the environment-scenery. We show that, $P \otimes Q$-almost surely, (ω, ξ) is a "good" environment-scenery.

In the following, we use c_i $(1 \le i \le 33)$ to denote finite and positive constants.

2 Preliminaries

In this section, we collect some basic properties of random walk in random environment that will be useful in the forthcoming sections.

2.1 About the environment

In the study of one-dimensional RWRE, an important role is played by a function of the environment ω, called the potential. This process, noted $V = (V(x),\ x \in \mathbb{Z})$, is defined on (Ω, P) by:

$$(2.1) \qquad V(x) := \begin{cases} \sum_{i=1}^{x} \log(\frac{1-\omega_i}{\omega_i}) & \text{if } x \ge 1, \\ 0 & \text{if } x = 0, \\ -\sum_{i=x+1}^{0} \log(\frac{1-\omega_i}{\omega_i}) & \text{if } x \le -1. \end{cases}$$

By (1.1), we observe that $|V(x) - V(x-1)| \le L$ for any $x \in \mathbb{Z}$. Moreover, we define $P_z\{\cdot\} := P\{\cdot \,|\, V(0) = z\}$, for any $z \in \mathbb{R}$; thus $P = P_0$. (Strictly speaking, we should

be working in a canonical space for V, with P_z defined as the image measure of P under translation.)

Let us define, for any Borel set $A \subset \mathbb{R}$,

$$\nu^+(A) \quad := \quad \min\{n \geq 0 : \, V(n) \in A\}.$$

We recall the following result, whose proof is given by a simple martingale argument.

Lemma 2.1 *For any $x < y < z$, we have*

$$\frac{y-x}{z-x+L} \leq P_y\{\nu^+([z,\infty)) < \nu^+((-\infty,x])\} \quad \leq \quad \frac{y-x+L}{z-x}.$$

Proof. Since (1.1) and (1.2) imply that $(V(n);\, n \geq 0)$ is a martingale with bounded jumps, we apply the optional stopping theorem ([7], p. 270) at $\nu^+([z,\infty)) \wedge \nu^+((-\infty,x])$ to get

$$\begin{aligned} y = E_y[X_0] \quad &= \quad E_y[X_{\nu^+([z,\infty))}\,;\, \nu^+([z,\infty)) < \nu^+((-\infty,x])] \\ &\qquad + E_y[X_{\nu^+((-\infty,x])}\,;\, \nu^+([z,\infty)) > \nu^+((-\infty,x])]. \end{aligned}$$

Since $X_{\nu^+([z,\infty))} \in [z, z+L]$ and $X_{\nu^+((-\infty,x])} \in [x-L, x]$ by ellipticity, we obtain

$$y \geq z P_y\{\nu^+([z,\infty)) < \nu^+((-\infty,x])\} + (x-L)(1 - P_y\{\nu^+([z,\infty)) < \nu^+((-\infty,x])\}),$$

which yields the right inequality. The left inequality is obtained by similar arguments. $\square$

Moreover, we recall a result of Hirsch [13], which, under assumptions (1.1)–(1.3), takes the following simplified form: for any $0 < \varepsilon' < \frac{1}{34}$, there exists $c_1 > 0$ such that

$$P\{\max_{0 \leq x \leq N} V(x) < N^{\frac{1}{2}-\varepsilon'}\} \sim c_1 N^{-\varepsilon'}, \qquad N \to \infty. \tag{2.2}$$

2.2 Quenched results

We define, for any $x \in \mathbb{Z}$,

$$\tau(x) := \min\{n \geq 1 : \, X_n = x\}, \qquad \min \emptyset := \infty.$$

(Note in particular that when $X_0 = x$, then $\tau(x)$ is the first *return* time to x.) Throughout the paper, we write $P_\omega^x\{\cdot\} := P_\omega\{\cdot \mid X_0 = x\}$ (thus $P_\omega^0 = P_\omega$) and denote by E_ω^x the expectation with respect to P_ω^x.

Recalling that $\omega_i/(1-\omega_i) = \mathrm{e}^{-(V(i)-V(i-1))}$, we get, for any $r < x < s$,

$$P_\omega^x\{\tau(r) < \tau(s)\} = \sum_{j=x}^{s-1} \mathrm{e}^{V(j)} \left(\sum_{j=r}^{s-1} \mathrm{e}^{V(j)} \right)^{-1} . \tag{2.3}$$

This result is proved in [25], see formula (2.1.4).

The next result, which gives a simple bound for the expectation of $\tau(r) \wedge \tau(s)$ when the walk starts from a site $x \in (r, s)$, is essentially contained in Golosov [9]; its proof can be found in [22]. For any integers $r < s$, we have

$$\max_{x \in (r,\, s) \cap \mathbb{Z}} E_\omega^x[\tau(s)\, \mathbf{1}_{\{\tau(s)<\tau(r)\}}] \le (s-r)^2 \exp\left[\max_{r \le i \le j \le s} (V(j) - V(i)) \right] . \tag{2.4}$$

We will also use the following estimate borrowed from Lemma 7 of Golosov [9]: for $\ell \ge 1$ and $x < y$,

$$P_\omega^x\{\tau(y) < \ell\} \le \ell \, \exp\left(- \max_{x \le i < y} [V(y-1) - V(i)] \right) . \tag{2.5}$$

Looking at the environment backwards, we get: for $\ell \ge 1$ and $w < x$,

$$P_\omega^x\{\tau(w) < \ell\} \le \ell \, \exp\left(- \max_{w < i \le x} [V(w+1) - V(i)] \right) . \tag{2.6}$$

Finally we quote an important result about excursions of Sinai's walk (for detailed discussions, see Section 3 of [6]). Let $b \in \mathbb{Z}$ and $x \in \mathbb{Z}$, and consider $L(\tau(b), x)$ under P_ω^b. In words, we look at the number of visits to the site x by the random walk (starting from b) until the first return to b. Then there exist constants c_2 and c_3 such that

$$c_2\, \mathrm{e}^{-[V(x)-V(b)]} \le E_\omega^b[L(\tau(b), x)] \le c_3\, \mathrm{e}^{-[V(x)-V(b)]}. \tag{2.7}$$

3 Good environment-scenery and proof of Theorem 1.1

For any $j \in \mathbb{N}^*$, we define

$$\begin{aligned} d^+(j) &:= \min\{n \ge 0 :\ V(n) \ge j\}, \\ b^+(j) &:= \min\left\{ n \ge 0 :\ V(n) = \min_{0 \le x \le d^+(j)} V(x) \right\}. \end{aligned}$$

These variables enable us to consider the valley $(0, b^+(j), d^+(j))$. Similarly, we define

$$\begin{aligned} d^-(j) &:= \max\{n \le 0 :\ V(n) \ge j\}, \\ b^-(j) &:= \max\left\{ n \le 0 :\ V(n) = \min_{d^-(j) \le x \le 0} V(x) \right\}. \end{aligned}$$

In the next sections, we will be frequently using the following elementary estimates.

Lemma 3.1 *For any $\varepsilon' > 0$, we have, P-almost surely for all large j,*

$$j^{2-\varepsilon'} \le |b^{\pm}(j)| < |d^{\pm}(j)| \le j^{2+\varepsilon'}.$$

Proof. Fix $\varepsilon' > 0$. Let us consider the sequence $(j_p)_{p\ge1}$ defined by $j_p := p^{12/\varepsilon'}$ for all $p \ge 1$. Using (2.2), we obtain $\sum_{p\ge1} P\{d^+(j_p) > \frac{1}{3}j_p^{2+\varepsilon'}\} < \infty$. Therefore, Borel-Cantelli lemma implies that, P-almost surely, $d^+(j_p) \le \frac{1}{3}j_p^{2+\varepsilon'}$ for all large p, say $p \ge p_0$. We fix a realization of ω and consider $j_p \le j \le j_{p+1}$ with $p \ge p_0$. Since $d^+(j) \le d^+(j_{p+1})$, we get

$$d^+(j) \le \frac{1}{3}j_{p+1}^{2+\varepsilon'} \le j^{2+\varepsilon'}\frac{1}{3}\left(\frac{j_{p+1}}{j}\right)^{2+\varepsilon'} \le j^{2+\varepsilon'}\frac{1}{3}\left(\frac{j_{p+1}}{j_p}\right)^{2+\varepsilon'} = j^{2+\varepsilon'}\frac{1}{3}(1+p^{-1})^{\frac{12(2+\varepsilon')}{\varepsilon'}},$$

which yields $d^+(j) \le j^{2+\varepsilon'}$ for all large j. In a similar way, we can prove that $j^{2-\varepsilon'} \le \nu^+((-\infty, -j^{1-\kappa}]) \le d^+(j)$ for some $\kappa > 0$ and all large j, which implies $j^{2-\varepsilon'} \le b^+(j)$ for all large j. Moreover, the arguments are the same to prove that, P-almost surely, $j^{2-\varepsilon'} \le |b^-(j)| < |d^-(j)| \le j^{2+\varepsilon'}$ for all large j. □

To introduce the announced "good" environment-scenery, we fix $\varepsilon > 0$ such that assumption of Part *(i)* of Theorem 1.1 holds. For $\alpha \in (0,1)$ (which will depend on ε), $0 < c_4 < 1/6$, and $j \ge 100$, we define

$$\begin{aligned} \gamma_0(j) &:= j, \\ \gamma_i(j) &:= j^{(1-\alpha)^i} = (\gamma_{i-1}(j))^{1-\alpha}, \qquad i \ge 1, && (3.1)\\ \varepsilon_i(j) &:= \exp\Big\{-c_4\gamma_{i+2}(j)\Big\}, \qquad i \ge 0. && (3.2) \end{aligned}$$

For convenience of notation we define $\varepsilon_{-1}(j) := \varepsilon_0(j)$. In words, $(\gamma_i(j))_{i\ge0}$ represents a decreasing sequence of distances, which enables us to classify the sites according to the value of $V(x) - V(b^+(j))$.

Write $\log_p$ for the p-th iterative logarithmic function. Fix $\varepsilon' := \min\{1/35, \varepsilon/2\} > 0$, and introduce, for $j \ge 100$,

$$M(j) := \inf\left\{n \ge 0 : \gamma_n(j) \le (\log_2 j)^{\frac{1-\alpha}{2+\varepsilon'}}\right\}. \qquad (3.3)$$

By definition of $M(j)$, we have

$$\gamma_{M(j)-1}(j) \in \left[(\log_2 j)^{\frac{1-\alpha}{2+\varepsilon'}}, (\log_2 j)^{\frac{1}{2+\varepsilon'}}\right].$$

Moreover, in view of (3.1) and since $\gamma_M(j)$ belongs to $[(\log_2 j)^{\frac{(1-\alpha)^2}{2+\varepsilon'}}, (\log_2 j)^{\frac{1-\alpha}{2+\varepsilon'}}]$, we get that

$$M(j) \sim \frac{1}{|\log(1-\alpha)|} \log_2 j, \qquad j \to \infty. \tag{3.4}$$

Note that we choose α small enough such that

$$\beta := (1-\alpha)^2 (2+\varepsilon) - (2+\varepsilon') > 0, \tag{3.5}$$

$$\beta' := \frac{\varepsilon'}{2} - \alpha > 0. \tag{3.6}$$

Then we introduce the set (the constant c_5 will be chosen small enough in (5.9))

$$\overline{\Theta}_{M(j)-1}(j) := \left[b^+(j) - c_5 \left(\gamma_{M(j)-1}(j)\right)^{2+\varepsilon'},\, b^+(j) + c_5 \left(\gamma_{M(j)-1}(j)\right)^{2+\varepsilon'}\right],$$

and, for $i = M(j)-2, ..., 1, 0$, the sets (the constant $c_6 \geq 1$ will be chosen large enough in (5.17))

$$\overline{\Theta}_i(j) := \left[b^+(j) - c_6 \left(\gamma_i(j)\right)^{2+\varepsilon'},\, b^+(j) + c_6 \left(\gamma_i(j)\right)^{2+\varepsilon'}\right] \setminus \bigcup_{p=i+1}^{M(j)-1} \overline{\Theta}_p(j).$$

Observe that the sets $(\overline{\Theta}_i(j))_{0\leq i\leq M(j)-1}$ form a partition of the interval $[b^+(j) - c_6\, j^{2+\varepsilon'},\, b^+(j) + c_6\, j^{2+\varepsilon'}]$. The final sets we consider are given, for $0 \leq i \leq M(j)-1$, by

$$\Theta_i(j) := \overline{\Theta}_i(j) \cap I(j),$$

where $I(j) := [\nu^+((-\infty,-j]), d^+(j)]$. Note that $\nu^+((-\infty,-j]) < d^+(j)$ on $A(j)$ which will be defined in (3.14). In this case, the sets $(\Theta_i(j))_{0\leq i\leq M(j)-1}$ form a partition of $I(j)$ into annuli (since $c_6 \geq 1$). Loosely speaking, the set $\Theta_i(j)$ contains the sites x satisfying $V(x) - V(b^+(j)) \approx \gamma_i(j)$. To cover $[d^-(j), d^+(j)]$, we define

$$\Theta_{-1}(j) := \left[-j^{2+\varepsilon'}, j^{2+\varepsilon'}\right] \cap \left[d^-(j), \nu^+((-\infty,-j])\right]. \tag{3.7}$$

Moreover, for the environment on $\mathbb{Z}^+$, we introduce the events

$$A_1^{env}(j) := \left\{-4j \leq V(b^+(j)) \leq -3j\right\}, \tag{3.8}$$

$$A_2^{env}(j) := \left\{\max_{0\leq x\leq y\leq b^+(j)} [V(y) - V(x)] \leq \frac{j}{4}\right\}. \tag{3.9}$$

The first event ensures that the valley considered is "deep enough" and the second one that the particle reaches the bottom of the valley "fast enough". To control the time spent by the particle in different $\Theta_i(j)$ during an excursion from $b^+(j)$ to $b^+(j)$, we define

$$(3.10)\quad A^{env}_{ann}(j) \;:=\; \bigcap_{i=0}^{M(j)-2} \left\{ \sum_{x\in\Theta_i(j)} \mathrm{e}^{-[V(x)-V(b^+(j))]} \le (\varepsilon_i(j))^2 \right\} =: \bigcap_{i=0}^{M(j)-2} A^{env}_{ann,i}(j).$$

For the environment on $\mathbb{Z}^-$, let

$$(3.11)\qquad B^{env}(j) \;:=\; \left\{ V(b^-(j)) \le -\frac{j}{6}\,,\ \max_{d^-(j)\le x\le y\le 0} [V(y)-V(x)] \le \frac{j}{3} \right\},$$

which ensures that the particle will not spent too much time on $\mathbb{Z}^-$.

Recalling that $\xi^-_x = \max\{-\xi_x, 0\}$, we define for the scenery

$$(3.12)\qquad A^{sce}_i(j) \;:=\; \left\{ \max_{x\in\Theta_i(j)} \xi^-_x < (\varepsilon_i(j))^{-1/2} \right\}, \qquad -1 \le i \le M(j)-2,$$

which ensures that the scenery does not reach excessive negative value in each $\Theta_i(j)$. In order to force the scenery in a neighborhood of the bottom (where the particle is concentrated), to be close to $a = \operatorname{ess\,sup} \xi_0$, we fix $\rho \in (0,1)$ and introduce

$$(3.13)\qquad A^{sce}_{M(j)-1}(j) \;:=\; \left\{ \min_{x\in\Theta_{M(j)-1}(j)} \xi_x \ge a-\rho \right\}.$$

We set

$$A^{env}(j) := A^{env}_1(j) \cap A^{env}_2(j) \cap A^{env}_{ann}(j), \qquad A^{sce}(j) := \bigcap_{i=-1}^{M(j)-1} A^{sce}_i(j).$$

Moreover, we define

$$(3.14)\qquad A(j) := A^{env}(j) \cap B^{env}(j) \cap A^{sce}(j).$$

A pair (ω,ξ) is a "good" environment-scenery if $(\omega,\xi) \in A(j)$ for infinitely many $j \in \mathbb{N}$.

For future use, let us note that for $\omega \in B^{env}(j) \cap A^{env}_2(j)$, we have

$$(3.15)\qquad \max_{d^-(j)\le x\le y\le b^+(j)} [V(y)-V(x)] \le \frac{2j}{3}.$$

To prove Theorem 1.1, we need two propositions, whose proofs are respectively postponed until Sections 5 and 4. The first one ensures that almost all pair (ω,ξ) is a "good" environment-scenery, while the second one describes the behavior of the particle in a "good" environment.

Proposition 3.2 *Under assumptions (1.1)–(1.3), we have that $P \otimes Q$-almost all (ω, ξ) is a "good" environment-scenery. More precisely, $P \otimes Q$-almost surely, there exists a random sequence $(m_k)_{k \geq 1}$ such that $m_k \geq k^{3k}$ and (ω, ξ) is a good environment-scenery along $(m_k)_{k\geq 1}$, i.e., $(\omega, \xi) \in A(m_k)$, for all $k \geq 1$.*

In fact $(m_k)_{k\geq 1}$ is constructed in the following way. Let us first introduce the sequence $j_p := p^{3p}$ for $p \geq 0$. We define then $(m_k)_{k\geq 1}$ by $m_1 := \inf\{j_p \geq 0 : (\omega, \xi) \in A(j_p)\}$ and $m_k := \inf\{j_p > m_{k-1} : (\omega, \xi) \in A(j_p)\}$ for $k \geq 2$. Then, Proposition 3.2 means that $m_k \to \infty$, $k \to \infty$, $P \otimes Q$-almost surely. Before establishing the proposition about the behavior of the particle, we extract a random sequence $(n_k)_{k\geq 1}$ from $(m_k)_{k\geq 1}$ such that

$$\sum_{k \geq 1} \varepsilon_{M(n_k)}(n_k) < \infty. \tag{3.16}$$

In fact, we consider the random sequence defined by $n_1 := \inf\{m_p \geq 1 : \varepsilon_{M(m_p)}(m_p) \leq 1\}$ and $n_k := \inf\{m_p > n_{k-1} : \varepsilon_{M(m_p)}(m_p) \leq \frac{1}{k^2}\}$ for $k \geq 2$.

To ease notations, we write throughout the paper, $d_k^+ := d^+(n_k)$, $\tau_k^+ := \tau(d_k^+)$, $b_k^+ := b^+(n_k)$ and $d_k^- := d^-(n_k)$, $\tau_k^- := \tau(d_k^-)$. Moreover, we define, for all $k \geq 1$,

$$t_k := \lfloor \mathrm{e}^{n_k} \rfloor. \tag{3.17}$$

Proposition 3.3 *For $P \otimes Q$ almost all (ω, ξ), we have that, P_ω-a.s., for all large k,*

$$L\left(t_k, \Theta_{-1}(n_k)\right) \leq \varepsilon_{-1}(n_k)\, t_k, \tag{3.18}$$

$$L\left(t_k, \Theta_i(n_k)\right) \leq \varepsilon_i(n_k)\, t_k, \qquad 0 \leq i \leq M(n_k) - 2, \tag{3.19}$$

$$\tau_k^+ \wedge \tau_k^- > t_k. \tag{3.20}$$

Remark 3.4 *There is no measurability problem for the events described in Proposition 3.3, see the beginning of Section 4. Similar arguments apply to the forthcoming events.*

Proof of Theorem 1.1.
Proof of Part (i). For any $\delta > 0$, we define $\varepsilon^{(\delta)}(j) := \sum_{i=-1}^{M(j)-2} \varepsilon_i^\delta(j)$. Recalling (1.6), we use Proposition 3.3 and Lemma 3.1 to obtain, for $\mathbb{P} \otimes Q$-almost all realization of ω, ξ and $(X_j)_{j\geq 0}$,

$$\sum_{j=0}^{t_k} \xi(X_j) \geq (1 - \varepsilon^{(1)}(n_k))\, t_k \left(\min_{x \in \Theta_{M(n_k)-1}(n_k)} \xi_x \right) - \sum_{i=-1}^{M(n_k)-2} \varepsilon_i(n_k)\, t_k \left(\max_{x \in \Theta_i(n_k)} \xi_x^- \right),$$

for all large k. Then, Proposition 3.2 implies

$$\begin{aligned} \sum_{j=0}^{t_k} \xi(X_j) &\geq (1-\varepsilon^{(1)}(n_k))\, t_k\, (a-\rho) - \sum_{i=-1}^{M(n_k)-2} \sqrt{\varepsilon_i(n_k)}\, t_k \\ &\geq (1-\varepsilon^{(1)}(n_k))\, t_k\, (a-\rho) - \varepsilon^{(1/2)}(n_k)\, t_k, \end{aligned} \tag{3.21}$$

for all large k. We claim that, for any $\delta > 0$ and all large j,

$$\varepsilon^{(\delta)}(j) \leq \sum_{i=-1}^{M(j)} \varepsilon_i^{\delta}(j) \leq 2\left(1+\frac{1}{\delta}\right) \varepsilon_{M(j)}^{\delta}(j). \tag{3.22}$$

To prove (3.22), we observe that

$$\sum_{i=-1}^{M(j)} \varepsilon_i^{\delta}(j) \leq 2\, \varepsilon_{M(j)}^{\delta}(j) + \sum_{i=0}^{M(j)-1} \int_{\varepsilon_i(j)}^{\varepsilon_{i+1}(j)} \frac{\varepsilon_i^{\delta}(j)}{\varepsilon_{i+1}(j)-\varepsilon_i(j)}\, \mathrm{d}x.$$

Recalling (3.2), we have that $\varepsilon_{i+1}(j)-\varepsilon_i(j) = \varepsilon_{i+1}(j)\left(1-\mathrm{e}^{-c_4(\gamma_{i+2}(j)-\gamma_{i+3}(j))}\right)$. Recalling (3.1) we get that $\gamma_{i+2}(j)-\gamma_{i+3}(j) = \gamma_{i+2}(j)(1-\gamma_{i+2}^{-\alpha}(j))$. Since (3.1) and (3.3) imply $\gamma_{i+2}(j) \geq \gamma_{M(j)+2}(j) \geq (\log_2 j)^{\frac{(1-\alpha)^4}{2+\varepsilon'}}$ for $0 \leq i \leq M(j)$, we obtain that $\gamma_{i+2}(j)-\gamma_{i+3}(j) \geq \gamma_{i+2}(j)/2$, for all large j and for $0 \leq i \leq M(j)$. Therefore, we get $\varepsilon_{i+1}(j)-\varepsilon_i(j) \geq \varepsilon_{i+1}(j)/2$, implying that

$$\varepsilon^{(\delta)}(j) \leq 2\, \varepsilon_{M(j)}^{\delta}(j) + 2 \sum_{i=0}^{M(j)-1} \int_{\varepsilon_i(j)}^{\varepsilon_{i+1}(j)} \frac{\varepsilon_i^{\delta}(j)}{\varepsilon_{i+1}(j)}\, \mathrm{d}x.$$

Furthermore, $\sum_{i=0}^{M(j)-1} \int_{\varepsilon_i(j)}^{\varepsilon_{i+1}(j)} \frac{\varepsilon_i^{\delta}(j)}{\varepsilon_{i+1}(j)}\, \mathrm{d}x \leq \sum_{i=0}^{M(j)-1} \int_{\varepsilon_i(j)}^{\varepsilon_{i+1}(j)} x^{\delta-1}\, \mathrm{d}x = \int_{\varepsilon_0(j)}^{\varepsilon_{M(j)}(j)} x^{\delta-1}\, \mathrm{d}x$, which is less than $\varepsilon_{M(j)}^{\delta}(j)/\delta$. This implies (3.22).

Combining (3.21) and (3.22) and recalling that $\varepsilon_{M(j)}^{\delta}(j) \to 0$ when $j \to \infty$, we get

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n} \xi(X_i) \geq a-\rho, \qquad \mathbb{P}\otimes Q\text{-a.s.} \tag{3.23}$$

To conclude the proof, it remains only to observe that (3.23) is true for all $\rho > 0$ and that the definition of a implies that $\mathbb{P}\otimes Q$-a.s., $\frac{1}{n}\sum_{i=0}^{n} \xi(X_i) \leq a$, for all $n \geq 0$. $\square$

Proof of Part (ii). Using Theorem 1.5 of [15], we have that, for any $\varepsilon'' > 0$, $\mathbb{P}$-almost surely, $\max_{0\leq i\leq n} X_i \geq (\log n)^{2-\varepsilon''} + 1$, for all large n. This implies

$$\sum_{i=0}^{n} \xi(X_i) \leq a\, n - \max_{0\leq x\leq \lceil (\log n)^{2-\varepsilon''}\rceil} \xi_x^{-}. \tag{3.24}$$

By assumption, there exists $\varepsilon > 0$ such that $Q\{\xi_0 < -\lambda\} \geq (\log \lambda)^{-2+\varepsilon}$. Therefore, fixing $\varepsilon'' < \varepsilon$, we get for $k \geq 1$ and all $N \geq 1$,

$$Q\left\{\max_{0\leq x\leq N} \xi_x^- < k\,a\,\mathrm{e}^{N^{\frac{1}{2-\varepsilon''}}}\right\} \leq \exp\left\{-c_7 N^\delta\right\}, \tag{3.25}$$

where $\delta := 1 - \frac{2-\varepsilon}{2-\varepsilon''} > 0$.

We choose $N_p := \lfloor (\log p)^T \rfloor$ for $p \geq 1$ with T large enough such that $T\delta > 1$. Therefore, (3.25) and the Borel–Cantelli lemma imply that, Q-almost surely, there exists $p_0(\xi)$ such that

$$\max_{0\leq x\leq N_p} \xi_x^- \geq k\,a\,\mathrm{e}^{N_p^{\frac{1}{2-\varepsilon''}}}, \tag{3.26}$$

for $p \geq p_0(\xi)$. Fixing a realization of ξ, we define $p(n)$ by

$$N_{p(n)} \leq \lceil (\log n)^{2-\varepsilon''} \rceil \leq N_{p(n)+1}, \tag{3.27}$$

for all n such that $p(n) \geq p_0(\xi)$. This yields

$$\max_{0\leq x\leq \lceil (\log n)^{2-\varepsilon''}\rceil} \xi_x^- \geq \max_{0\leq x\leq N_{p(n)}} \xi_x^- \geq k\,a\,\mathrm{e}^{N_{p(n)}^{\frac{1}{2-\varepsilon''}}},$$

the last inequality being a consequence of (3.26). Therefore, we obtain

$$\begin{aligned}
\max_{0\leq x\leq \lceil (\log n)^{2-\varepsilon''}\rceil} \xi_x^- &\geq ka \exp\left\{\lceil (\log n)^{2-\varepsilon''}\rceil^{\frac{1}{2-\varepsilon''}}\right\} \exp\left\{-\left(\lceil (\log n)^{2-\varepsilon''}\rceil^{\frac{1}{2-\varepsilon''}} - N_{p(n)}^{\frac{1}{2-\varepsilon''}}\right)\right\} \\
&\geq kan \exp\left\{-\left(N_{p(n)+1}^{\frac{1}{2-\varepsilon''}} - N_{p(n)}^{\frac{1}{2-\varepsilon''}}\right)\right\},
\end{aligned}$$

the second inequality being a consequence of (3.27). Moreover, we easily get that $N_{p(n)+1}^{\frac{1}{2-\varepsilon''}} - N_{p(n)}^{\frac{1}{2-\varepsilon''}} \to 0$, when $n \to \infty$, implying that for all large n,

$$\max_{0\leq x\leq \lceil (\log n)^{2-\varepsilon''}\rceil} \xi_x^- \geq \frac{k}{2}\,an. \tag{3.28}$$

Assembling (3.24) and (3.28), we get that $\mathbb{P}\otimes Q$-almost surely, $\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^n \xi(X_i) \leq a\,(1-\frac{k}{2})$. We conclude the proof by sending k to infinity. □

Remark 3.5 *It is possible to give more precision in the case (ii). Indeed, using the same arguments, we can prove that if* $Q\{\xi_0^- > \lambda\} \geq \frac{1}{(\log\lambda)^\alpha}$, *for some* $\alpha < 2$, *then we have, for any* $\varepsilon' > 0$, *that* $\lim_{n\to\infty} n^{-\frac{2}{\alpha}+\varepsilon'} Z_n = -\infty$, $\mathbb{P}\otimes Q$*-almost surely.*

4 Proof of Proposition 3.3

Let us first explain why the events described in Proposition 3.3 (more precisely in (3.18) and (3.19)) are measurable. Since the sequences $(n_k)_{k\geq 0}$ and $(t_k)_{k\geq 0}$ can be explicitly constructed, $\omega \mapsto (n_k)_{k\geq 0}(\omega)$ and $\omega \mapsto (t_k)_{k\geq 0}(\omega)$ are measurable. Moreover, this implies that $\Theta_i(n_k)$ is measurable, for any $-1 \leq i \leq M(n_k) - 2$. Now, let us write

$$L(t_k, \Theta_i(n_k)) = \sum_{x\in\mathbb{Z}} L(t_k, x)\mathbf{1}_{\{x\in\Theta_i(n_k)\}}.$$

Since the $\Theta_i(n_k)$'s are measurable, so are the random variables $\mathbf{1}_{\{x\in\Theta_i(n_k)\}}$. To the other hand, the measurability of $L(t_k, x)$, for any $x \in \mathbb{Z}$, is obvious, being the composition of the measurable applications $\omega \mapsto (t_k)_{k\geq 0}(\omega)$ and $t \mapsto L(t, x)$.

We now proceed to the proof of Proposition 3.3. To get (3.20), we observe that

$$P_\omega\left\{\tau_k^+ \wedge \tau_k^- \leq t_k\right\} \leq P_\omega\left\{\tau_k^+ \leq t_k\right\} + P_\omega\left\{\tau_k^- \leq t_k\right\}.$$

Then using (2.5), (2.6) and (3.8), (3.11) we obtain

$$P_\omega\left\{\tau_k^+ \wedge \tau_k^- \leq t_k\right\} \leq t_k\,(\mathrm{e}^{-4n_k} + \mathrm{e}^{-7n_k/6}) \leq 2\,\mathrm{e}^{-n_k/6},$$

Since $n_k \geq k$, this yields

$$\sum_{k\geq 0} P_\omega\left\{\tau_k^+ \wedge \tau_k^- \leq t_k\right\} \leq 2\sum_{k\geq 0} \mathrm{e}^{-n_k/6} < \infty.$$

We conclude by using the Borel–Cantelli lemma.

To prove (3.19), we apply the strong Markov property at $\tau(b_k^+)$ and get for $0 \leq i \leq M(n_k) - 2$,

$$\begin{aligned} & P_\omega\left\{L\left(t_k, \Theta_i(n_k)\right) \geq \varepsilon_i(n_k)\, t_k\right\} \\ \leq\ & P_\omega^{b_k^+}\left\{L\left(t_k, \Theta_i(n_k)\right) \geq \varepsilon_i(n_k)\, t_k - \lambda_k\right\} + P_\omega\left\{\lambda_k \leq \tau(b_k^+) \leq \tau_k^-\right\} + P_\omega\left\{\tau_k^- \leq \tau(b_k^+)\right\}, \end{aligned}$$

for any $\lambda_k \geq 0$. By (2.3), (3.9) and Lemma 3.1, we get, for all large k,

$$P_\omega\left\{\tau_k^- \leq \tau(b_k^+)\right\} \leq \frac{b_k^+\,\mathrm{e}^{n_k/3}}{\mathrm{e}^{n_k}} \leq n_k^{2+\varepsilon'}\,\mathrm{e}^{-2n_k/3}.$$

Since $P_\omega\left\{\lambda_k \leq \tau(b_k^+) \leq \tau_k^-\right\} \leq \lambda_k^{-1} E_\omega\left[\tau(b_k^+)\,\mathbf{1}_{\{\tau(b_k^+)\leq\tau_k^-\}}\right]$, (2.4) and (3.15) yield

$$P_\omega\left\{\lambda_k \leq \tau(b_k^+) \leq \tau_k^-\right\} \leq \frac{(b_k^+ - d_k^-)^2}{\lambda_k}\,\mathrm{e}^{2n_k/3} \leq \frac{2n_k^{2\,(2+\varepsilon')}}{\lambda_k}\,\mathrm{e}^{2n_k/3},$$

for all large k, the second inequality being a consequence of Lemma 3.1. Choosing $\lambda_k := \mathrm{e}^{5n_k/6}$, we obtain, for all large k,

$$P_\omega\left\{\lambda_k \le \tau(b_k^+) \le \tau_k^-\right\} + P_\omega\left\{\tau_k^- \le \tau(b_k^+)\right\} \le \mathrm{e}^{-n_k/7}. \tag{4.1}$$

To treat $P_{k,i} := P_\omega^{b_k^+}\left\{L\left(t_k, \Theta_i(n_k)\right) \ge \varepsilon_i(n_k)\, t_k - \lambda_k\right\}$, we observe that (3.17) implies $\lambda_k \le 2\,\mathrm{e}^{-n_k/6}\, t_k$. Therefore, we obtain

$$P_{k,i} \;\; \le \;\; P_\omega^{b_k^+}\left\{L\left(t_k, \Theta_i(n_k)\right) \ge \left(\varepsilon_i(n_k) - 2\,\mathrm{e}^{-n_k/6}\right) t_k\right\}.$$

Then, by Chebyshev's inequality, we get

$$P_{k,i} \;\; \le \;\; \frac{1}{\left(\varepsilon_i(n_k) - 2\,\mathrm{e}^{-n_k/6}\right) t_k}\, E_\omega^{b_k^+}\left[L\left(t_k, \Theta_i(n_k)\right)\right].$$

Furthermore, observe that Sinai's walk can not make more than t_k excursions from b_k^+ to b_k^+ before t_k. Since these excursions are i.i.d., we obtain

$$P_{k,i} \;\; \le \;\; \frac{t_k}{\left(\varepsilon_i(n_k) - 2\,\mathrm{e}^{-n_k/6}\right) t_k}\, E_\omega^{b_k^+}\left[L\left(\tau(b_k^+), \Theta_i(n_k)\right)\right].$$

Now we recall (2.7), which implies $E_\omega^{b_k^+}\left[L\left(\tau(b_k^+), \Theta_i(n_k)\right)\right] \le c_3 \sum_{x\in\Theta_i(n_k)} \mathrm{e}^{-[V(x)-V(b_k^+)]}$, for all $0 \le i \le M(n_k)-2$. Moreover, by (3.10), we get for all large k and for $0 \le i \le M(n_k)-2$,

$$P_{k,i} \le \frac{c_3\, \left(\varepsilon_i(n_k)\right)^2}{\left(\varepsilon_i(n_k) - 2\,\mathrm{e}^{-n_k/6}\right)} \le c_8\, \varepsilon_i(n_k),$$

for some $c_8 > 0$. The second inequality is a consequence of $\varepsilon_i(n_k) \ge \varepsilon_0(n_k)$ and the fact that $c_4 < 1/6$ implies $\mathrm{e}^{-n_k/6} = o(\varepsilon_0(n_k))$.

Summing from 0 to $M(n_k)-2$ and using (3.22), we get, with $c_9 := 2(1+\frac{1}{\delta})\, c_8$,

$$\sum_{i=0}^{M(n_k)-2} P_{k,i} \le c_8 \sum_{i=0}^{M(n_k)-2} \varepsilon_i(n_k) \le c_9\, \varepsilon_{M(n_k)}(n_k). \tag{4.2}$$

Assembling (4.1), (4.2) and recalling (3.4), (3.16) we obtain

$$\sum_{k\ge1} \sum_{i=0}^{M(n_k)-2} P_\omega\left\{L\left(t_k, \Theta_i(n_k)\right) \ge \varepsilon_i(n_k)\, t_k\right\} \le \sum_{k\ge1}\left(c_9\, \varepsilon_{M(n_k)}(n_k) + M(n_k)\,\mathrm{e}^{-n_k/7}\right) < \infty.$$

This implies (3.19) by an application of the Borel–Cantelli lemma.

We get (3.18) by an argument very similar to the one used in the proof of (3.19), the main ingredients being the facts that $V(x) - V(b_k^+) \ge 2\,n_k$, for $x \in \Theta_{-1}(n_k)$ (which is a consequence of (3.11), (3.8) and the definition of $d^+(j)$), and that $\Theta_{-1}(n_k)$ contains less than $2\,n_k^{2+\varepsilon'}$ sites (by (3.7)). We feel free to omit the details.

5 Proof of Proposition 3.2

We now prove that, for $P \otimes Q$-almost all (ω, ξ), there exists a sequence (m_k) such that $(\omega, \xi) \in A(m_k)$, $\forall k \geq 1$, where $A(m_k)$ is defined in (3.14).

Let $j_k := k^{3k}$ $(k \geq 1)$ and $\mathcal{F}_{j_{k-1}} := \sigma\{V(x), \xi_z,\, d^-(j_{k-1}) \leq x, z \leq d^+(j_{k-1})\}$. In the following, we ease notations by using γ_i, ε_i and M instead of $\gamma_i(j_k)$, $\varepsilon_i(j_k)$ and $M(j_k)$.

If we are able to show that

$$(5.1) \qquad \sum_k P \otimes Q \left\{ A(j_k) \,|\, \mathcal{F}_{j_{k-1}} \right\} = \infty, \qquad P \otimes Q\text{-a.s.},$$

then Lévy's Borel–Cantelli lemma ([7], p. 237) will tell us that $P \otimes Q$-almost surely there are infinitely many k such that $(\omega, \xi) \in A(j_k)$.

To bound $P \otimes Q\{A(j_k) \,|\, \mathcal{F}_{j_{k-1}}\}$ from below, we start with the trivial inequality $A(j_k) \supset A(j_k) \cap C(j_{k-1})$, for any set $C(j_{k-1})$. We choose $C(j_{k-1}) := C^{env}(j_{k-1}) \cap D^{env}(j_{k-1}) \cap C^{sce}(j_{k-1})$, where

$$\begin{aligned} C^{env}(j_{k-1}) &:= \left\{ \inf_{0 \leq y \leq d^+(j_{k-1})} V(y) \geq -j_{k-1} \log^2 j_{k-1} \right\}, \\ D^{env}(j_{k-1}) &:= \left\{ \inf_{d^-(j_{k-1}) \leq y \leq 0} V(y) \geq -j_{k-1} \log^2 j_{k-1} \right\}, \\ C^{sce}(j_{k-1}) &:= \left\{ \max_{d^-(j_{k-1}) \leq x \leq d^+(j_{k-1})} \xi_x^- < (\varepsilon_{-1}(j_k))^{-1/2} \right\}. \end{aligned}$$

Clearly, $C(j_{k-1})$ is $\mathcal{F}_{j_{k-1}}$-measurable. Moreover on $C^{env}(j_{k-1}) \cap A^{env}(j_k)$, we have $d^+(j_{k-1}) \leq \nu^+((-\infty, -j_k]) \leq b^+(j_k)$.

Let

$$E_{-1}^{sce}(j_k) := \left\{ \max_{x \in \Theta_{-1} \setminus [d^-(j_{k-1}), d^+(j_{k-1})]} \xi_x^- < (\varepsilon_{-1}(j_k))^{-1/2} \right\},$$

and consider

$$E^{sce}(j_k) := \bigcap_{i=0}^{M-1} A_i^{sce}(j_k) \cap E_{-1}^{sce}(j_k).$$

Since $C^{sce}(j_{k-1}) \cap E_{-1}^{sce}(j_k) \subset A_{-1}^{sce}(j_k)$, it follows that

$$\begin{aligned} & P \otimes Q \left\{ A(j_k) \,|\, \mathcal{F}_{j_{k-1}} \right\} \\ \geq\ & P \otimes Q \left\{ P \otimes Q \left\{ A^{env}(j_k)\,,\, B^{env}(j_k)\,,\, E^{sce}(j_k)\,,\, C(j_{k-1}) \,|\, \mathcal{F}_{j_{k-1}}, \omega \right\} \,|\, \mathcal{F}_{j_{k-1}} \right\} \\ \geq\ & P \otimes Q \left\{ \mathbf{1}_{\{A^{env}(j_k)\,,\ B^{env}(j_k)\,, C(j_{k-1})\}} P \otimes Q \left\{ E^{sce}(j_k) \,|\, \omega \right\} \,|\, \mathcal{F}_{j_{k-1}} \right\}. \end{aligned}$$

Now, we suppose for the moment that we are able to prove that there exists $c_{10} > 0$ such that, for P-almost all ω,

$$P \otimes Q \left\{ E^{sce}(j_k) \,|\, \omega \right\} \geq \frac{c_{10}}{k^{1/4}}. \tag{5.2}$$

We get

$$\begin{aligned} P \otimes Q \left\{ A(j_k) \,|\, \mathcal{F}_{j_{k-1}} \right\} &\geq \frac{c_{10}}{k^{1/4}} \, P \otimes Q \left\{ A^{env}(j_k) \,,\; B^{env}(j_k) \,,\, C(j_{k-1}) \,|\, \mathcal{F}_{j_{k-1}} \right\} \\ &\geq \frac{c_{10}}{k^{1/4}} \, P_k^+ \, P_k^- \, \mathbf{1}_{C^{sce}(j_{k-1})}, \end{aligned} \tag{5.3}$$

where we use the fact that $(V(x),\, x \geq 0)$ and $(V(x),\, x < 0)$ are independent processes and introduce

$$\begin{aligned} P_k^+ &:= P \left\{ A^{env}(j_k) \,,\; C^{env}(j_{k-1}) \,|\, \mathcal{F}_{j_{k-1}} \right\}, \\ P_k^- &:= P \left\{ B^{env}(j_k) \,,\; D^{env}(j_{k-1}) \,|\, \mathcal{F}_{j_{k-1}} \right\}. \end{aligned}$$

To bound P_k^+ from below, we introduce

$$E_2^{env}(j_k) \;:=\; \left\{ \max_{0 \leq x \leq y \leq b^+(j_k)} [V(y) - V(x)] \leq \frac{j_k}{4} - j_{k-1} \log^2 j_{k-1} - j_{k-1} - L \right\},$$

and consider

$$E^{env}(j_k) := A_1^{env}(j_k) \cap A_{ann}^{env}(j_k) \cap E_2^{env}(j_k).$$

Observe that $C^{env}(j_{k-1}) \cap \{\max_{d^+(j_{k-1}) \leq x \leq y \leq b^+(j_k)} [V(y) - V(x)] \leq \frac{j_k}{4} - j_{k-1} \log^2 j_{k-1} - j_{k-1} - L\} \subset A_2^{env}(j_k)$. Thus, since $V(d^+(j_{k-1})) \in I_{j_{k-1}} := [j_{k-1},\, j_{k-1} + L]$, we have, by applying the strong Markov property at $d^+(j_{k-1})$,

$$P_k^+ \geq \left(\inf_{z \in I_{j_{k-1}}} P_z \left\{ E^{env}(j_k) \right\} \right) \mathbf{1}_{C^{env}(j_{k-1})}. \tag{5.4}$$

To bound P_k^- from below, we observe the following inclusion

$$B^{env}(j_k) \supset \left\{ \max_{d^-(j_k) \leq x \leq y \leq d^-(j_{k-1})} [V(y) - V(x)] \leq \frac{j_k}{3} \right\} \cap D^{env}(j_{k-1}).$$

Then since $V(d^-(j_{k-1}))$ belongs to $I_{j_{k-1}}$, the strong Markov property applied at $d^-(j_{k-1})$ yields

$$P_k^- \geq \left(\inf_{z \in I_{j_{k-1}}} P_z \left\{ B^{env}(j_k) \right\} \right) \mathbf{1}_{D^{env}(j_{k-1})}. \tag{5.5}$$

Observe that an easy calculation yields $\mathbf{1}_{C(j_{k-1})} = 1$, $P \otimes Q$-almost surely for all large k. Therefore, recalling (5.3), (5.4) and (5.5), the proof of (5.1) boils down to showing that

$$\liminf_{k\to\infty} \inf_{z\in I_{j_{k-1}}} P_z \{E^{env}(j_k)\} > 0, \tag{5.6}$$

$$\liminf_{k\to\infty} \inf_{z\in I_{j_{k-1}}} P_z \{B^{env}(j_k)\} > 0. \tag{5.7}$$

The rest of the section is devoted to the proof of (5.2) and (5.6), whereas (5.7) is an immediate consequence of Donsker's theorem.

5.1 Proof of (5.2)

Since the sets $\{\Theta_i\}_{-1\le i\le M-1}$ are disjoint, the events $E^{sce}_{-1}(j_k)$ and $\{A^{sce}_i(j_k)\}_{0\le i\le M-1}$ are mutually independent. We write

$$P \otimes Q \{E^{sce}(j_k) \,|\, \omega\} = \prod_{i=0}^{M-1} P \otimes Q \{A^{sce}_i(j_k) \,|\, \omega\} \times P \otimes Q \left\{E^{sce}_{-1}(j_k) \,|\, \omega\right\}.$$

Thus, (5.2) will be a consequence of the two following lemmas.

Lemma 5.1 *For P-almost all ω, we have*

$$P \otimes Q \left\{A^{sce}_{M-1}(j_k) \,|\, \omega\right\} \ge \frac{1}{k^{1/4}}.$$

Lemma 5.2 *There exists $c_{11} > 0$ such that, for P-almost all ω,*

$$\liminf_{k\to\infty} \prod_{i=0}^{M-2} P \otimes Q \{A^{sce}_i(j_k) \,|\, \omega\} \times P \otimes Q \left\{E^{sce}_{-1}(j_k) \,|\, \omega\right\} \ge c_{11}. \tag{5.8}$$

Proof of Lemma 5.1. Recalling (3.13), (3.7) and (3.4), we get, P-almost surely,

$$P \otimes Q \left\{A^{sce}_{M-1}(j_k) \,|\, \omega\right\} \ge \exp\{c_5 \log q \, \log_2 j_k\},$$

where $q := Q\{\xi_0 \ge a - \rho\}$. Note that the definition of a implies $-\infty < \log q < 0$. Therefore, it remains only to observe that $\log_2 j_k = \log k + \log_2 k + \log 3$ and to choose c_5 small enough such that

$$c_5 \log q > -1/5, \tag{5.9}$$

to conclude the proof. $\square$

Proof of Lemma 5.2. Recalling (3.12) and that $(\xi_x^-)_{x\in\mathbb{Z}}$ is a family of i.i.d. random variables, we get, P-almost surely, for $0 \le i \le M-2$,

$$\begin{aligned} P\otimes Q\,\{A_i^{sce}(j_k)\,|\,\omega\} &\ge \Big(Q\{\xi_0^- \le \varepsilon_i^{-1/2}\}\Big)^{2c_6\,\gamma_i^{2+\varepsilon'}} \\ &\ge \exp\Big\{2c_6\,\gamma_i^{2+\varepsilon'}\,\log\Big(1-Q\{\xi_0^-\ge\varepsilon_i^{-1/2}\}\Big)\Big\}. \end{aligned}$$

Then, since $Q\{\xi_0^- \ge \varepsilon_i^{-1/2}\}$ tends to 0 when k tends to ∞ and using the fact that $\log(1-x) \ge -c_{12}\,x$ for $x\in[0,1/2)$ with $c_{12} := 2\log 2 > 0$, it follows that

$$P\otimes Q\,\{A_i^{sce}(j_k)\,|\,\omega\} \ge \exp\Big\{-c_{13}\,\gamma_i^{2+\varepsilon'}\,Q\{\xi_0^-\ge\varepsilon_i^{-1/2}\}\Big\},$$

for all large k, with $c_{13} := 2\,c_6\,c_{12}$. Recalling that $Q\left\{\xi_0^- \ge \lambda\right\} \le (\log\lambda)^{-(2+\varepsilon)}$ for $\lambda \ge \lambda_0 > 0$ and (3.2) we get for k large enough and uniformly in $0\le i\le M-2$,

$$P\otimes Q\,\{A_i^{sce}(j_k)\,|\,\omega\} \ge \exp\Big\{-c_{14}\,\gamma_i^{-\beta}\Big\}, \tag{5.10}$$

where $\beta := (1-\alpha)^2\,(2+\varepsilon) - (2+\ \varepsilon') > 0$ by (3.5), and $c_{14} := c_{13}\left(\frac{2}{c_4}\right)^{2+\varepsilon}$. Similarly, since $E_{-1}^{sce}(j_k) \subset A_{-1}^{sce}(j_k)$ and recalling (3.12), we obtain

$$P\otimes Q\,\left\{E_{-1}^{sce}(j_k)\,|\,\omega\right\} \ge \exp\Big\{-c_{15}\,\gamma_0^{-\beta}\Big\}, \tag{5.11}$$

for some $c_{15} > 0$. Combining (5.10) and (5.11), we get

$$\prod_{i=0}^{M-2} P\otimes Q\,\{A_i^{sce}(j_k)\,|\,\omega\} \times P\otimes Q\,\left\{E_{-1}^{sce}(j_k)\,|\,\omega\right\} \ge \exp\{-c_{16}\,\sigma_\beta\},$$

with $c_{16} := \max\{c_{14}, c_{15}\}$ and $\sigma_\beta := \gamma_0^{-\beta} + \sum_{i=0}^{M-2}\gamma_i^{-\beta}$. By the same way we proved (3.22), we obtain that, for any $\beta > 0$, there exists $c_{17} \le 1 + 2/\beta$ such that $\sigma_\beta \le c_{17}\gamma_{M-1}^{-\beta}$. Recalling (3.4), it follows that $\sigma_\beta \to 0$ when $k\to\infty$, which implies (5.8). $\square$

5.2 Proof of (5.6)

To prove (5.6), we need the following preliminary result.

Lemma 5.3 *For any $\delta > 0$, $k \geq 1$ and any $0 \leq p \leq M$, we have*

$$\sum_{i=p}^{M} \gamma_i^\delta \leq \left(1 + \frac{2}{\delta}\right) \gamma_p^\delta.$$

Proof. Observe that we easily get

$$\sum_{i=p}^{M} \gamma_i^\delta \leq \gamma_p^\delta + \sum_{i=p+1}^{M} \int_{\gamma_i}^{\gamma_{i-1}} \frac{\gamma_i^\delta}{\gamma_{i-1} - \gamma_i} dx.$$

Recalling that $\gamma_{i-1} - \gamma_i \geq \gamma_{i-1}/2$, for all large j and for $1 \leq i \leq M$, we get

$$\sum_{i=p}^{M} \gamma_i^\delta \leq \gamma_p^\delta + 2 \sum_{i=p+1}^{M} \int_{\gamma_i}^{\gamma_{i-1}} \frac{\gamma_i^\delta}{\gamma_{i-1}} dx.$$

Then, $\sum_{i=p+1}^{M} \int_{\gamma_i}^{\gamma_{i-1}} \frac{\gamma_i^\delta}{\gamma_{i-1}} dx \leq \sum_{i=p+1}^{M} \int_{\gamma_i}^{\gamma_{i-1}} x^{\delta-1} dx = \int_{\gamma_M}^{\gamma_p} x^{\delta-1} dx \leq \gamma_p^\delta/\delta$ yields Lemma 5.3.
$\square$

We now proceed to prove (5.6). Let

$$a_\ell := -3 j_k - \ell\, \gamma_M, \qquad F_1^{env}(j_k, \ell) := \left\{ a_{\ell+1} \leq V(b^+(j_k)) < a_\ell \right\}.$$

Denoting $\theta_k := \lfloor j_k/\gamma_M \rfloor - 1$, the inclusion $\bigsqcup_{\ell=0}^{\theta_k} F_1^{env}(j_k, \ell) \subset A_1^{env}(j_k)$ yields

$$P_z\{E^{env}(j_k)\} \;\; \geq \;\; \sum_{\ell=0}^{\theta_k} P_z\{F_1^{env}(j_k, \ell)\,,\, A_{ann}^{env}(j_k)\,,\, E_2^{env}(j_k)\} =: \sum_{\ell=0}^{\theta_k} P_{k,\ell}^+. \tag{5.12}$$

To bound $P_{k,\ell}^+$ by below for $0 \leq \ell \leq \theta_k$, we define the following levels,

$$\eta_i = \eta_i(j_k, \ell) \;\; := \;\; a_\ell + \gamma_i, \qquad 0 \leq i \leq M, \tag{5.13}$$

$$\eta_{M+1} = \eta_{M+1}(j_k, \ell) \;\; := \;\; a_\ell, \tag{5.14}$$

$$\eta_{M+2} = \eta_{M+2}(j_k, \ell) \;\; := \;\; a_{\ell+1}. \tag{5.15}$$

In the following, we introduce stopping times for the potential, which will enable us to consider a valley having "good" properties. Let us write

$$\begin{aligned} T = T(j_k, \ell) \;\; &:= \;\; \nu^+((-\infty, \eta_{M+1}]), \\ \tilde{T} = \tilde{T}(j_k, \ell) \;\; &:= \;\; \nu^+((-\infty, \eta_{M+2}]). \end{aligned}$$

Then, let us define the following stopping times, for $0 \le i \le M$,

$$\begin{aligned} T_i = T_i(j_k,\ell) &:= \nu^+((-\infty,\eta_i]), \\ T_i' = T_i'(j_k,\ell) &:= \min\{n \ge T : V(n) \ge \eta_{M-i}\}, \\ R_i = R_i(j_k,\ell) &:= \min\{n \ge T_i' : V(n) \le \eta_{M-i+1}\}. \end{aligned}$$

We introduce the events

$$\begin{aligned} G_i(j_k) &:= \left\{T_{i+1} - T_i \le \gamma_i^{2+\varepsilon'}, \max_{T_i \le x \le y \le T_{i+1}} [V(y) - V(x)] \le \frac{j_k}{5}\right\}, \qquad 0 \le i \le M-1, \\ G_M(j_k) &:= \left\{T - T_M \le \gamma_M^{2+\varepsilon'}, \max_{T_M \le x \le y \le T} [V(y) - V(x)] \le \frac{j_k}{5}\right\}, \end{aligned}$$

and

$$\begin{aligned} G_0'(j_k) &:= \left\{T_0' - T \le \gamma_M^{2+\varepsilon'}, T_0' < \tilde{T}\right\}, \\ G_i'(j_k) &:= \left\{T_i' - T_{i-1}' \le \gamma_{M-i}^{2+\varepsilon'}, T_i' < R_{i-1}\right\}, \qquad 1 \le i \le M. \end{aligned}$$

Moreover, we set

$$G(j_k,\ell) := \bigcap_{i=0}^{M} G_i(j_k), \qquad G'(j_k,\ell) := \bigcap_{i=0}^{M} G_i'(j_k),$$

and

$$\begin{aligned} H(j_k,\ell) &:= \left\{\max_{0 \le x \le y \le T_0} [V(y) - V(x)] \le \frac{j_k}{5}\right\}, \\ H'(j_k,\ell) &:= \left\{d^+(j_k) < R_M\right\}. \end{aligned}$$

See Figure 1 for an example of $\omega \in G(j_k,\ell) \cap G'(j_k,\ell) \cap H(j_k,\ell) \cap H'(j_k,\ell)$.

Observe that on $G(j_k,\ell) \cap G'(j_k,\ell) \cap H'(j_k,\ell)$, we have, for $0 \le i \le M-1$,

$$[T_i, T_{M-i}'] \supset \left\{x \in [0, d^+(j_k)] : V(x) - V(b^+(j_k)) \le \gamma_{i+1}\right\}. \tag{5.16}$$

Moreover, on $G(j_k,\ell) \cap G'(j_k,\ell)$,

$$T_{M-i}' - T_i \le 2\sum_{p=i}^{M} \gamma_p^{2+\varepsilon'}, \qquad 0 \le i \le M.$$

If we choose c_6 such that

$$c_6 \ge 2\left(1 + \frac{2}{2+\varepsilon'}\right), \tag{5.17}$$

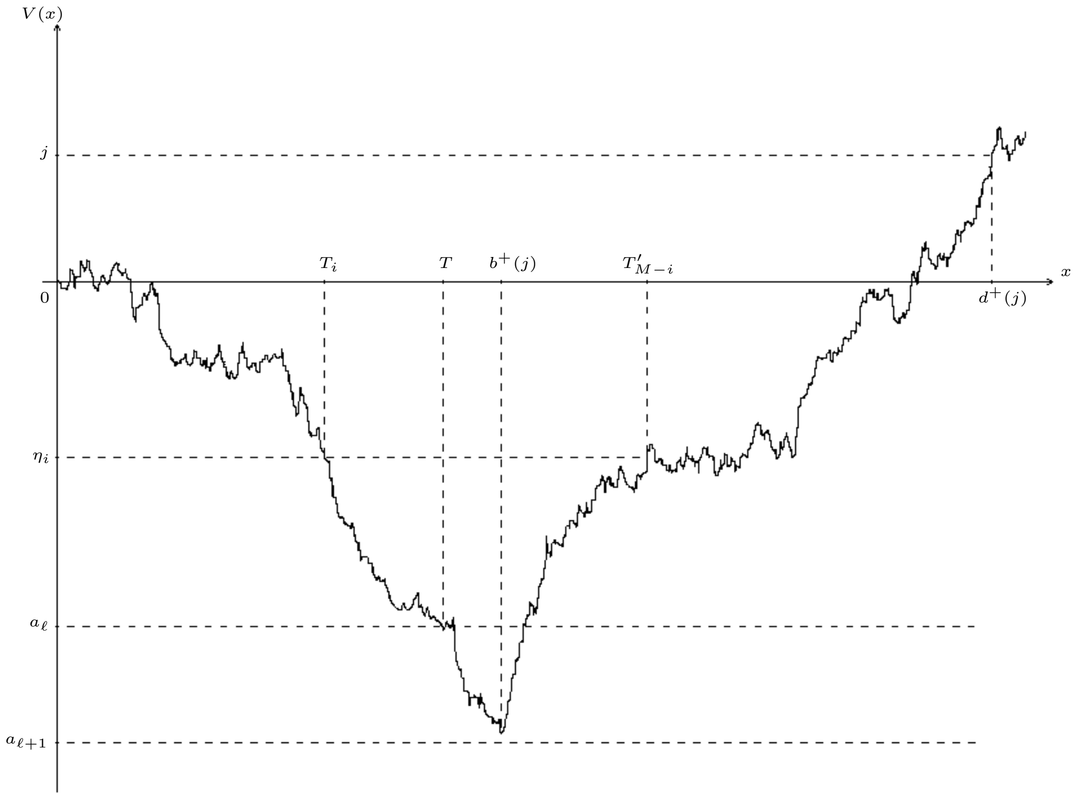

Figure 1: Example of $\omega \in G(j_k,\ell) \cap G'(j_k,\ell) \cap H(j_k,\ell) \cap H'(j_k,\ell)$

then Lemma 5.3 yields

$$(5.18) \qquad [T_i, T'_{M-i}] \subset [b^+(j_k) - c_6\, \gamma_i^{2+\varepsilon'},\, b^+(j_k) + c_6\, \gamma_i^{2+\varepsilon'}], \qquad 0 \le i \le M-2.$$

Recall now definition of $\Theta_i(j_k)$, so that, by assembling (5.16) and (5.18), we have on $G(j_k,\ell) \cap G'(j_k,\ell) \cap H'(j_k,\ell)$,

$$\Theta_i(j_k) \subset \left\{ x \in \mathbb{Z} :\, V(x) - V(b^+(j_k)) \ge\, \gamma_{i+2} \right\}, \qquad 0 \le i \le M-2.$$

An easy calculation yields $\sum_{x \in \Theta_i(j_k)} \exp\{-[V(x) - V(b^+(j_k))]\} \,\le\, 2c_6\, \gamma_i^{2+\varepsilon'} \mathrm{e}^{-\gamma_{i+2}}$, for all $0 \le i \le M-2$, on $G(j_k,\ell) \cap G'(j_k,\ell) \cap H'(j_k,\ell)$. On the other hand, since $6\, c_4 < 1$, we get $2c_6\, \gamma_i^{2+\varepsilon'} \mathrm{e}^{-\gamma_{i+2}} \le \varepsilon_i^2$, for all large k and uniformly in $0 \le i \le M-2$. This implies that $G(j_k,\ell) \cap G'(j_k,\ell) \cap H'(j_k,\ell) \subset F_1^{env}(j_k,\ell) \cap A_{ann}^{env}(j_k)$. We easily observe that $G(j_k,\ell) \cap H(j_k,\ell) \subset E_2^{env}(j_k)$, for all large k. Thus we obtain

$$G(j_k,\ell) \cap G'(j_k,\ell) \cap H(j_k,\ell) \cap H'(j_k,\ell) \subset F_1^{env}(j_k,\ell) \cap A_{ann}^{env}(j_k) \cap E_2^{env}(j_k).$$

Recalling (5.12), we get

$$P^+_{k,\ell} \geq P_z\left\{G(j_k,\ell)\,,\, G'(j_k,\ell)\,,\, H(j_k,\ell)\,,\, H'(j_k,\ell)\right\}.$$

To bound $P_z\{G(j_k,\ell)\,,\, G'(j_k,\ell)\,,\, H(j_k,\ell)\,,\, H'(j_k,\ell)\}$ by below, we will apply the strong Markov property several times.

Since $V(T'_M) \in I_{\eta_0} := [\eta_0,\, \eta_0 + L]$, the strong Markov property applied at T'_M implies, for $z \in I_{j_{k-1}}$,

$$P^+_{k,\ell} \geq P_z\Big\{G(j_k,\ell)\,,\, G'(j_k,\ell)\,,\, H(j_k,\ell)\Big\} \inf_{y\in I_{\eta_0}} P_y\left\{d^+(j_k) \leq \nu^+((-\infty,\eta_1])\right\}.$$

To bound by below $P_y\{\cdots\}$ on the right hand side, observe that $P_y\{\cdots\}$ is greater than $P_{\eta_0}\{\cdots\}$. Moreover since $\eta_1 \geq -4j_k$ implies $j_k - \eta_1 \leq 5j_k$, Lemma 2.1 yields

$$P_{\eta_0}\{d^+(j_k) \leq \nu^+((-\infty,\eta_1])\} \geq \frac{\eta_0 - \eta_1}{5j_k + L} = \frac{\eta_0(1-\eta_0^{-\alpha})}{5j_k + L} \geq c_{18},$$

for all large k and some $c_{18} > 0$, which implies

$$P^+_{k,\ell} \geq c_{18}\, P_z\left\{G(j_k,\ell)\,,\, G'(j_k,\ell)\,,\, H(j_k,\ell)\right\}.$$

We now apply the strong Markov property successively at $(T'_{M-i})_{1\leq i\leq M}$ and T, such that

$$P^+_{k,\ell} \geq c_{18}\, P_z\left\{G(j_k,\ell)\,,\, H(j_k,\ell)\right\} \inf_{y\in I_{\eta_{M+1}-L}} Q'_{0,y} \prod_{p=1}^{M} \inf_{y\in I_{\eta_{M-p+1}}} Q'_{p,y}, \tag{5.19}$$

where

$$Q'_{p,y} \;:=\; P_y\left\{d^+(\eta_{M-p}) \leq \gamma_{M-p}^{2+\varepsilon'}\,,\, d^+(\eta_{M-p}) < \nu^+((-\infty,\eta_{M-p+2}])\right\}, \qquad 0 \leq p \leq M.$$

First, observe that $\inf_{y\in I_{\eta_{M-p+1}}} Q'_{p,y} \geq Q'_{p,\eta_{M-p+1}} =: Q'_p$, for all $1 \leq p \leq M$ and similarly $\inf_{y\in I_{\eta_{M+1}-L}} Q'_{0,y} \geq Q'_{0,\eta_{M+1}-L} := Q'_0$. Therefore we only have to bound from below Q'_p for $1 \leq p \leq M$ and Q'_0. Recalling that $P\{A\,,\, B\} \geq P\{A\} - P\{B^c\}$, we get, for $1 \leq p \leq M$,

$$Q'_p \geq P_{\eta_{M-p+1}}\left\{d^+(\eta_{M-p}) < \nu^+((-\infty,\eta_{M-p+2}])\right\} - P_{\eta_{M-p+1}}\left\{d^+(\eta_{M-p}) \geq \gamma_{M-p}^{2+\varepsilon'}\right\},$$

and

$$Q'_0 \geq P_{\eta_{M+1}-L}\left\{d^+(\eta_M) < \nu^+((-\infty,\eta_{M+2}])\right\} - P_{\eta_{M+1}-L}\left\{d^+(\eta_M) \geq \gamma_M^{2+\varepsilon'}\right\}.$$

By Lemma 2.1, we obtain, for $1 \le p \le M$,

$$P_{\eta_{M-p+1}}\left\{d^+(\eta_{M-p}) < \nu^+((-\infty, \eta_{M-p+2}])\right\} \ge \frac{\eta_{M-p+1} - \eta_{M-p+2}}{\eta_{M-p} - \eta_{M-p+2} + L},$$

and

$$P_{\eta_{M+1}-L}\left\{d^+(\eta_M) < \nu^+((-\infty, \eta_{M+2}])\right\} \ge \frac{\eta_{M+1} - L - \eta_{M+2}}{\eta_M - \eta_{M+2} + L}.$$

Recalling (5.13) and (5.14), we bound by below $P_{\eta_{M-p+1}}\{d^+(\eta_{M-p}) < \nu^+((-\infty, \eta_{M-p+2}])\}$ (for all $1 \le p \le M$) by

$$\begin{aligned}
\frac{\gamma_{M-p+1}}{\gamma_{M-p}} \frac{1 - \gamma_{M-p}^{-\alpha(1-\alpha)}}{1 + L\gamma_{M-p}^{-1}} &\ge \frac{\gamma_{M-p+1}}{\gamma_{M-p}}(1 - \gamma_{M-p}^{-\alpha(1-\alpha)})(1 - L\gamma_{M-p}^{-1}) \\
&\ge \frac{\gamma_{M-p+1}}{\gamma_{M-p}}(1 - 2\,\gamma_{M-p}^{-\alpha(1-\alpha)}),
\end{aligned}$$

for all large k. The first inequality is a consequence of $(1+x)^{-1} \ge 1 - x$ for any $x \in (0,1)$ and the second one is a consequence of $0 < \alpha < 1$. Similarly, recalling (5.13), (5.14) and (5.15), we get, for all large k,

$$P_{\eta_{M+1}-L}\left\{d^+(\eta_M) < \nu^+((-\infty, \eta_{M+2}])\right\} \ge \frac{\gamma_M - L}{2\,\gamma_M + L} \ge c_{18},$$

with $c_{18} > 0$. Moreover, combining (2.2) and the fact that $\gamma_{M-p} \le \gamma_{M-p}^{(2+\varepsilon')(\frac{1}{2} - \frac{\varepsilon'}{6})}$ for $0 \le p \le M$ yields

$$\begin{aligned}
P_{\eta_{M-p+1}}\left\{d^+(\eta_{M-p}) \ge \gamma_{M-p}^{2+\varepsilon'}\right\} &\le c_{19}\gamma_{M-p}^{-\varepsilon'/6}, \qquad 1 \le p \le M, \\
P_{\eta_{M+1}-L}\left\{d^+(\eta_M) \ge \gamma_M^{2+\varepsilon'}\right\} &\le c_{19}\gamma_M^{-\varepsilon'/6},
\end{aligned}$$

for all large k and for some $c_{19} > 0$. Therefore, we obtain $Q'_0 \ge c_{20}$ for some $c_{20} > 0$ and recalling (3.6) we get, for $1 \le p \le M$,

$$Q'_p \ge \frac{\gamma_{M-p+1}}{\gamma_{M-p}}(1 - c_{21}\gamma_{M-p}^{-\beta''}),$$

where $\beta'' := \min\{\alpha(1-\alpha), \beta'\} > 0$ (β' is defined in (3.6)) and $c_{21} > 0$. Observe that $\gamma_{M-p}^{-\beta''} \le \gamma_M^{-\beta''}$ for $1 \le p \le M$, and that $\gamma_M^{-\beta''} \to 0$, $k \to \infty$. Recalling the fact that $\log(1-x) \ge -c_{12}\,x$, for $x \in [0, 1/2)$, we obtain

$$\inf_{y \in I_{\eta_{M+1}-L}} Q'_{0,y} \prod_{p=1}^{M} \inf_{y \in I_{\eta_{M-p+1}}} Q'_{p,y} \ge c_{20}\frac{\gamma_M}{\gamma_0} \exp\left\{-c_{22}\sum_{p=1}^{M} \gamma_{M-p}^{-\beta''}\right\},$$

where $c_{22} := c_{12}c_{21}$.

Recall that for any $\beta'' > 0$, there exists $c_{23} > 0$ such that $\sum_{p=1}^{M} \gamma_{M-p}^{-\beta''} \le c_{23}\, \gamma_M^{-\beta''}$. Then, recalling (5.19), this yields, for all large k,

$$P_{k,\ell}^{+} \ge c_{24}\, \frac{\gamma_M}{\gamma_0}\, P_z \left\{ G(j_k, \ell)\, ,\, H(j_k, \ell) \right\}, \tag{5.20}$$

with $c_{24} > 0$. To bound $P_z\{G(j_k, \ell)\, ,\, H(j_k, \ell)\}$ from below, we apply successively the strong Markov property at $(T_{M-i})_{0 \le i \le M}$ such that

$$P_z \left\{ G(j_k, \ell)\, ,\, H(j_k, \ell) \right\} \ge P_z \left\{ H(j_k, \ell) \right\} \prod_{p=0}^{M} Q_p,$$

where

$$Q_p \;\; := \;\; P_{\eta_p} \left\{ \nu^{+}((-\infty, \eta_{p+1}]) \le \min \left\{ d^{+}(\eta_{p+1} + j_k/5), \gamma_p^{2+\varepsilon'} \right\} \right\}, \qquad 0 \le p \le M.$$

Recall that $P\{A\, ,\, B\} \ge P\{A\} - P\{B^c\}$. Then (2.2) yields, for $1 \le p \le M$,

$$P_{\eta_p} \left\{ \nu^{+}((-\infty, \eta_{p+1}]) \le \gamma_p^{2+\varepsilon'} \right\} \ge 1 - c_{25} \gamma_p^{-\varepsilon'/6},$$

with $c_{25} > 0$. Moreover, using Lemma 2.1, we get, for $1 \le p \le M$,

$$P_{\eta_p} \left\{ d^{+}(\eta_{p+1} + j_k/5) \le \nu^{+}((-\infty, \eta_{p+1}]) \right\} \le c_{26}\, \frac{\gamma_p}{j_k},$$

with $c_{26} > 0$. Therefore, observing that, for $1 \le p \le M$, we have $\frac{\gamma_p}{j_k} \le \frac{\gamma_1}{j_k} = j_k^{-\alpha} \to 0$, $k \to \infty$, and using the fact that $\log(1-x) \ge -c_{12}\, x$, for $x \in [0, 1/2)$, we get that

$$\prod_{p=1}^{M} Q_p \ge \exp \left\{ -c_{27} \sum_{p=1}^{M} \left(\gamma_p^{-\varepsilon'/6} + \frac{\gamma_p}{j_k} \right) \right\}, \tag{5.21}$$

where $c_{27} := c_{12} \max\{c_{25}, c_{26}\}$.

Recalling that $\sum_{p=1}^{M} \gamma_p^{-\varepsilon'/6} \le c_{28}\, \gamma_M^{-\varepsilon'/6}$ and $\sum_{p=1}^{M} \gamma_p \le c_{29}\, \gamma_1 = o(j_k)$ for some $c_{28}, c_{29} > 0$, (5.21) yields

$$P_z \left\{ G(j_k, \ell)\, ,\, H(j_k, \ell) \right\} \ge c_{30}\, Q_0\, P_z \left\{ H(j_k, \ell) \right\}, \tag{5.22}$$

for some $c_{30} > 0$. Observe that Donsker's theorem implies that there exists $c_{31} > 0$ such that $\min\{P_z\{H(j_k, \ell)\}, Q_0\} \ge c_{31}$. Therefore, assembling (5.20) and (5.22), we get

$$P_{k,\ell}^{+} \ge c_{32}\, \frac{\gamma_M}{\gamma_0},$$

where $c_{32} := c_{24}\, c_{30}\, c_{31}^2$.

Recalling (5.12) and $\theta_k = \lfloor j_k/\gamma_M \rfloor - 1$, we get, uniformly in $z \in I_{j_{k-1}}$,

$$P_z\left\{E^{env}(j_k)\right\} \geq c_{32}\, \theta_k \,\frac{\gamma_M}{\gamma_0} \geq c_{33},$$

for all large k and for some $c_{33} > 0$, which concludes the proof of (5.6). □

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References

[1] Andreoletti, P. (2005). Almost sure estimates for the concentration neighborhood of Sinai's walk. To appear in *Stochastic Processes and their Applications.* `ArXiv math.PR/0501439`.

[2] Asselah, A. and Castell, F. (2007). Random walk in random scenery and self-intersection local times in dimensions $d \geq 5$. *Probab. Theory Related Fields* **138**, 1–32.

[3] Bolthausen, E. (1989). A central limit theorem for two-dimensional random walks in random sceneries. *Ann. Probab.* **17**, 108–115.

[4] Borodin A. N. (1979). Limit theorems for sums of independent random variables defined on a transient random walk. *Investigations in the theory of probability distributions*, IV. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **85** , 17–29, 237, 244.

[5] Borodin A. N. (1979). A limit theorem for sums of independent random variables defined on a recurrent random walk. *Dokl. Akad. Nauk SSSR* **246:4**, 786–787.

[6] Dembo, A., Gantert, N., Peres, Y. and Shi, Z. (2007). Valleys and the maximum local time for random walk in random environment. *Probab. Theory Related Fields* **137**, 443–473.

[7] Durrett, R. (2004). *Probability: Theory and Examples* (third edition). Duxbury, Belmont.

[8] Gantert, N. and Shi, Z. (2002). Many visits to a single site by a transient random walk in random environment. *Stoch. Proc. Appl.* **99**, 159–176.

[9] Golosov, A.O. (1984). Localization of random walks in one-dimensional random environments. *Comm. Math. Phys.* **92**, 491–506.

[10] Golosov, A.O. (1986). Limit distributions for random walks in random environments. *Soviet Math. Dokl.* **28**, 18–22.

[11] Häggström, O. (1997). Infinite clusters in dependent automorphism invariant percolation on trees. *Ann. Probab.* **25**, 1423–1436.

[12] Häggström, O. and Peres, Y. (1997). Monotonicity of uniqueness for percolation on Cayley graphs: all infinite clusters are born simultaneously. *Probab. Theory Related Fields* **113**, 273–285.

[13] Hirsch, W. M.(1965). A strong law for the maximum cumulative sum of independent random variables. *Comm. Pure Appl. Math.* **18**, 109–127.

[14] den Hollander, F. and Steif, J. (2006). Random walk in random scenery: A survey of some recent results. Dynamics and Stochastics: Festschrift in Honor of Michael Keane. *IMS Lecture Notes-Monograph Series.* **48**, 53–65.

[15] Hu, Y. and Shi, Z. (1998). The limits of Sinai's simple random walk in random environment. *Ann. Probab.* **26**, 1477–1521.

[16] Hu, Y. and Shi, Z. (1998). The local time of simple random walk in random environment. *J. Theoret. Probab.* **11**, 765–793.

[17] Kesten, H. (1979). The limit distribution of Sinai's random walk in random environment. *Physica* **138A**, 299–309.

[18] Kesten, H. and Spitzer, F. (1979). A limit theorem related to a new class of self-similar processes. *Z. Wahrsch. Verw. Gebiete* **50**, 5–25.

[19] Lyons, R. and Schramm, O. (1999). Stationary measures for random walks in a random environment with random scenery. *New York J. Math.* **5**, 107–113.

[20] Révész, P. (2005). *Random Walk in Random and Non-Random Environments* (second edition). World Scientific, Singapore.

[21] Shi, Z. (1998). A local time curiosity in random environment. *Stoch. Proc. Appl.* **76**, 231–250.

[22] Shi, Z. and Zindy, O. (2007). A weakness in strong localization for Sinai's walk. *Ann. Probab.* **35**, 1118–1140.

[23] Sinai, Ya.G. (1982). The limiting behavior of a one-dimensional random walk in a random medium. *Th. Probab. Appl.* **27**, 256–268.

[24] Solomon, F. (1975). Random walks in a random environment. *Ann. Probab.* **3**, 1–31.

[25] Zeitouni, O. (2004). *Random Walks in Random Environment*, XXXI summer school in probability, St Flour (2001), *Lecture Notes in Math.* **1837**, pp. 193–312. Springer, Berlin.

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